MATH 5B - TEST 1 Spring 2024

(Chapter 6, excluding 6.7)

100 POINTS 97 oops 3 free points

Show ALL work with clear explanation of process . No credit will be given for solutions if work is not shown (except on the first ten problems where it is not necessary to show work).

**One page of formulas is allowed

Fill in the blanks. (2 points each)

(1) $\ln\left(e^{\cos x}\right) =$ cosx

(2) $\lim_{x\to\infty} e^{-x} =$ 0

(3) $\cos^{-1}\left(\frac{-\sqrt{2}}{2}\right) =$ $\frac{3\pi}{4}$ (exact)

(4) $\lim_{x\to 0^+} \ln x =$ $-\infty$

(5) $\cos(\sin^{-1}(-4/5)) =$ $3/5$

CIRCLE T FOR TRUE, F FOR FALSE.(2 pts each)

T F (6) $\frac{d}{dx}\log_5 x = \frac{1}{x\ln 5}$.

T F (7) $\frac{d}{dx}\left[4^x\right] = 4^x$

T F (8) $\tan^{-1}\left(\tan(\frac{3\pi}{4})\right) = \frac{-\pi}{4}$.

T F (9) $\int e^{7x}\,dx = \frac{1}{7}e^{7x} + C$

T F (10) $e^{\sin(x)\ln(x)} = x^{\sin(x)}$

(11) Given $f(x) = x^3 + 3\sin x + 4\cos x$,

(a) Find $\left(f^{-1}\right)'(4)$ (5 points)

$f^{-1}(4) = 0$

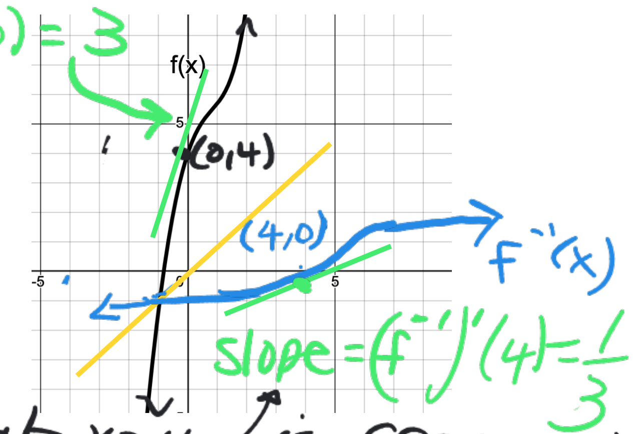
$f'(x) = 3x^2 + 3\cos x - 4\sin x$

$\left(f^{-1}\right)'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(0)} = \boxed{\frac{1}{3}}$

Work on presentation. Show steps in such a way someone could follow your thought process.

(b). Given the graph of $f(x)$, sketch the graph of $f^{-1}(x)$ and use it to explain if why your answer in part a is is reasonable. (2 points)

slope $= f'(0) = 3$



slope $= (f^{-1})'(4) = \frac{1}{3}$

The graph of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ across $y=x$. $(0,4)$ is on f, $(4,0)$ is on $f^{-1}(x)$. Looking at the graph, a slope of $1/3$ for tangent to $f^{-1}(x)$ at $x=4$ is reasonable.

(12) Find the derivative of each of the following functions and simplify: (4 pts each)

(a) $y = \ln\left(\frac{e^x}{\sqrt{x^2+1}}\right)$

use log properties to simplify before differentiating

there is an easy approach

$$y = \ln e^x - \ln(x^2+1)^{1/2}$$

$$y = x - \frac{1}{2}\ln(x^2+1)$$

$$y' = 1 - \frac{1}{2}\frac{1}{x^2+1}\cdot 2x$$

$$\boxed{y' = 1 - \frac{x}{x^2+1}}$$

can also do part (a) directly:

$$y' = \frac{1}{\frac{e^x}{\sqrt{x^2+1}}}\frac{d}{dx}\left(\frac{e^x}{\sqrt{x^2+1}}\right) = \frac{\sqrt{x^2+1}}{e^x}\frac{d}{dx}\left(\frac{e^x}{\sqrt{x^2+1}}\right)$$

quotient

$$= \frac{\sqrt{x^2+1}}{\cancel{e^x}}\cdot\frac{\sqrt{x^2+1}\,\cancel{e^x} - \cancel{e^x}\,\frac{1}{2}(x^2+1)^{-1/2}(2x)}{x^2+1}$$

$$= \frac{x^2+1-x}{x^2+1}$$

(b) $y = \ln(4x)\left(\cos^{-1}x\right)$

$$y' = \frac{4}{4x}\cos^{-1}x + \ln(4x)\cdot\frac{-1}{\sqrt{1-x^2}}$$

$$\boxed{y' = \frac{\cos^{-1}x}{x} - \frac{\ln(4x)}{\sqrt{1-x^2}}}$$

(c) $y = e^{\tan^{-1}(5x)}$

$$y' = e^{\tan^{-1}(5x)}\frac{d}{dx}\left(\tan^{-1}(5x)\right)$$

$$y' = e^{\tan^{-1}(5x)}\frac{1}{1+(5x)^2}\frac{d}{dx}5x$$

$$\boxed{y' = \frac{5e^{\tan^{-1}(5x)}}{1+25x^2}}$$

(d) $y = x^{\sqrt{x}}$

logarithmic differentiation

$$\ln y = \ln x^{\sqrt{x}}$$

$$\ln y = \sqrt{x}\ln x$$

$$\frac{1}{y}y' = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\cdot\frac{1}{x}$$

$$y' = y\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$$

$$y' = x^{\sqrt{x}}\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$$

$$\boxed{y' = x^{\sqrt{x}}\left(\frac{\ln x + 2}{2\sqrt{x}}\right)}$$

(13). Find $\frac{dy}{dx}$ if $y^2 = \ln(x^3 + y)$. (6 points)

implict differentiation

$$\frac{d}{dx}\, y^2 = \frac{d}{dx} \ln(x^3+y)$$

$$2yy' = \frac{1}{x^3+y} \frac{d}{dx}(x^3+y)$$

$$2yy' = \frac{1}{x^3+y}(3x^2+y')$$

$$(x^3+y)2yy' = 3x^2+y'$$

solve for y'

$$2x^3yy' + 2y^2y' = 3x^2 + y'$$

$$2x^3yy' + 2y^2y' - y' = 3x^2$$

$$y'(2x^3y+2y^2-1) = 3x^2$$

$$\boxed{y' = \frac{3x^2}{2x^3y+2y^2-1}}$$

(14) Find local extrema values for $f(x) = x\ln x$ Classify as max or min. Show how you know it is a max/min. (6 points)

Find critical numbers:

$$f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$f'(x) = 0 \Rightarrow \ln x + 1 = 0$$

$$x = e^{-1}$$

Adding just a few words clarifies explanation and shows understanding

First derivative test

$$\underset{\searrow}{-} \quad \underset{1/e}{|} \quad \underset{\nearrow}{+}$$

local min at $\left(\frac{1}{e}, -\frac{1}{e}\right)$

(15) Find the value of each of the following integrals. (simplify answers exactly) (5 pts each)

(a) $\int e^x \sqrt{2e^x - 1}\, dx$

$u = 2e^x - 1$

$du = 2e^x dx$

$\int e^x \sqrt{2e^x - 1}\, dx = \frac{1}{2}\int u^{1/2}\, du$

$= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C$

$= \boxed{\frac{1}{3}(2e^x - 1)^{3/2} + C}$

(b) $\int_0^{1/6} \frac{2}{\sqrt{1-9x^2}}\, dx$

$u = 3x$

$du = 3dx$

$\frac{1}{3}\int_0^{1/2} \frac{2}{\sqrt{1-u^2}}\, du$

$\frac{2}{3} \sin^{-1} u \Big]_0^{1/2}$

$\frac{2}{3} \sin^{-1} \frac{1}{2} = \frac{2}{3} \cdot \frac{\pi}{6}$

$\boxed{\frac{\pi}{9}}$

(c) $\int \frac{3}{x\sqrt{4x^2 - 4}}\, dx$

$= \frac{3}{2}\int \frac{1}{x\sqrt{x^2-1}}\, dx$

$= \boxed{\frac{3}{2} \sec^{-1} x + C}$

(d) $\int \frac{\cos x}{1 + \sin^2 x}\, dx$

$u = \sin x$

$du = \cos x\, dx$

$\int \frac{\cos x}{1 + \sin^2 x}\, dx = \int \frac{1}{1+u^2}\, du$

$= \tan^{-1} u + C$

$= \boxed{\tan^{-1}(\sin x) + C}$

Can check indefinite integrals by differentiating

(15 (cont'd)

(e) $\int \frac{\cos(\ln x)}{x}\,dx$

$u = \ln x$

$du = \frac{1}{x}dx$

$= \int \cos u\,du$

$= \sin u + c$

$= \boxed{\sin(\ln x) + c}$

(f) $\int \frac{1}{3x+7}\,dx$

$u = 3x+7$

$du = 3dx$

$= \frac{1}{3}\int \frac{1}{u}\,du$

$= \frac{1}{3}\ln|u| + c$

$= \boxed{\frac{1}{3}\ln|3x+7| + c}$

(16) Find each of the following limits. Calculus steps must be shown and correct notation must be used: (4 pts each)

(a) $\lim_{x\to 0} \frac{\sin x - x}{x^3}$ $\quad$ "$\frac{0}{0}$"

should not be used with = symbol

Apply L'Hospital's rule

$\stackrel{L}{=} \lim_{x\to 0} \frac{\cos x - 1}{3x^2}$ $\quad$ "$\frac{0}{0}$"

$\stackrel{L}{=} \lim_{x\to 0} \frac{-\sin x}{6x}$ $\quad$ "$\frac{0}{0}$"

$\stackrel{L}{=} \lim_{x\to 0} \frac{-\cos x}{6} = \boxed{-\frac{1}{6}}$

written at each step

(16 cont'd)

(b) $\lim_{x\to 0^+}\left(x^2 \ln x\right)$ "$0\cdot\infty$"

$$= \lim_{x\to 0^+} \frac{\ln x}{1/x^2} \quad \text{"}\frac{-\infty}{\infty}\text{"}$$

L'Hospital's

$$\stackrel{L}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x\to 0^+} \frac{1}{x}\cdot\frac{-x^3}{2} = \lim_{x\to 0} \frac{-x^2}{2} = \boxed{0}$$

simplify

(c) $\lim_{x\to 0^+} x^{\sin x}$ $= \lim_{x\to 0^+} e^{\sin x \ln x}$

0^0

$$= e^{\lim_{x\to 0^+} \sin x \ln x}$$

$$\lim_{x\to 0^+} \sin x \ln x \quad 0\cdot-\infty$$

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}}$$

$$\stackrel{L}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}}$$

$$= \lim_{x\to 0^+} \frac{-\sin^2 x}{x\cos x}$$

$$\stackrel{L}{=} \lim_{x\to 0^+} \frac{-2\sin x\cos x}{\cos x - x\sin x}$$

$$= 0$$

$$= e^0$$

$$= \boxed{1}$$